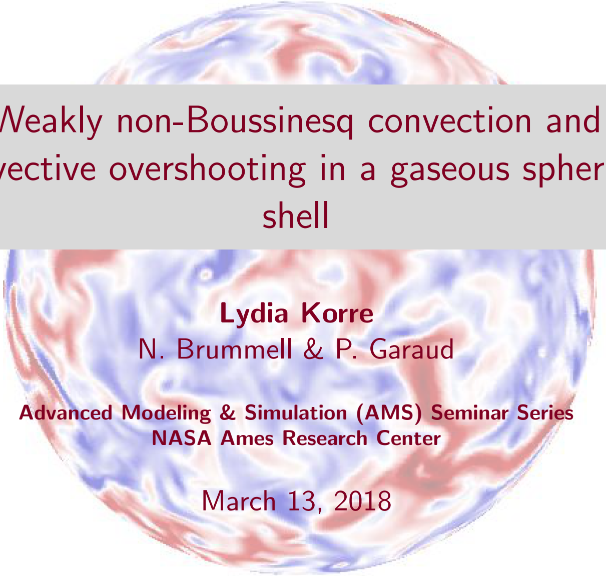

Weakly non-Boussinesq convection and convective overshooting in a gaseous spherical shell

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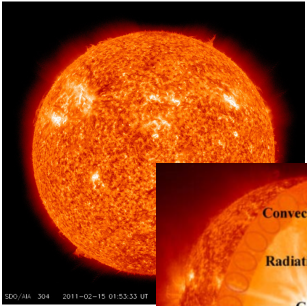
Motivation

Solar-type stars with outer convection zones

Image of the Sun (SDO gallery).



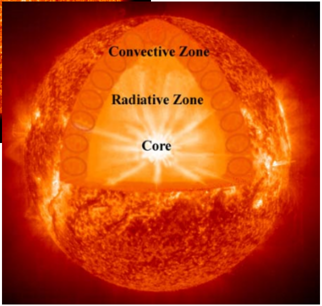
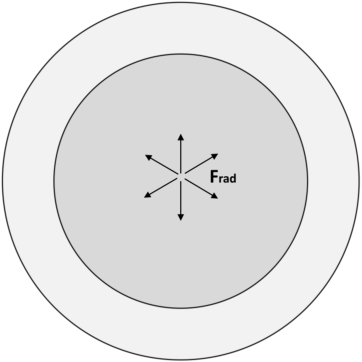


Image credit to ESA/NASA SOHO.

- Solar-type stars have thin outer convection zones (CZ) lying on top of a stable radiative zone (RZ).
- Nuclear burning in the core provides fixed flux of energy that must be transported to the surface.

Spherical shell geometry



- The simplest possible model is two concentric spherical shells with fixed flux coming through the inner boundary.
- Two cases studied:
  - convection only
  - convective overshooting

Part I

Weakly non-Boussinesq convection in a gaseous spherical shell

Dimensional SV Boussinesq equations * in a gaseous spherical shell

Let $T = T_{\mathrm{rad}}(r) + \Theta(r, \theta, \phi, t)$, then:

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_m} \nabla p + \alpha \Theta g \boldsymbol{e_r} + \nu \nabla^2 \boldsymbol{u},$$

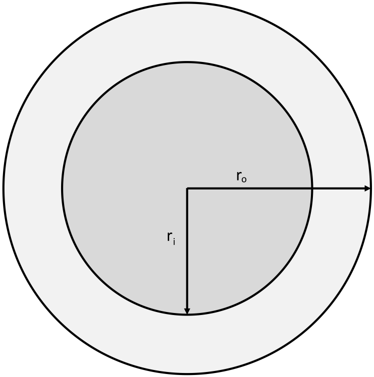
$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta + u_r \left(\frac{dT_{\mathrm{rad}}}{dr} - \boxed{\frac{dT_{\mathrm{ad}}}{dr}} \right) = \kappa \nabla^2 \Theta,$$

and

$$\rho / \rho_m = -\alpha \Theta, \quad \text{and} \quad dT_{\mathrm{ad}} / dr = -g / c_p$$

* Spiegel and Veronis, Astrophys. J. 131, 442 (1960)

Radiative temperature gradient



$$-4\pi r^2 \kappa \frac{dT_{\rm rad}}{dr} = L_\star \Rightarrow$$

$$\boxed{\frac{dT_{\rm rad}}{dr} \propto \frac{1}{r^2}}$$

Non-dimensional Boussinesq equations in a gaseous spherical shell

We then non-dimensionalize the problem by using the outer radius $[l] = r_o$ as the lengthscale, $[t] = r_o^2/\nu$ as the timescale, $[u] = \nu/r_o$ as the velocity scale and $[T] = |dT_{\mathrm{rad}}/dr - dT_{\mathrm{ad}}/dr|\Big|_{r=r_o} r_o$ as the temperature scale.

The non-dimensional equations are:

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{\mathrm{Ra}_o}{\mathrm{Pr}} \Theta \boldsymbol{e_r} + \nabla^2 \boldsymbol{u},$$

and

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta + \boxed{\beta(r)} u_r = \frac{1}{\mathrm{Pr}} \nabla^2 \Theta.$$

Non-dimensional quantities

- the Rayleigh number and the Rayleigh function

$$\mathrm{Ra}_o = \frac{\alpha g \left[\left|\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr}\right|\right]_{r=r_o} r_o^4}{\nu\kappa}, \mathrm{Ra}(r) = \frac{\alpha g \left|\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr}\right| r_o^4}{\nu\kappa}$$

- the Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

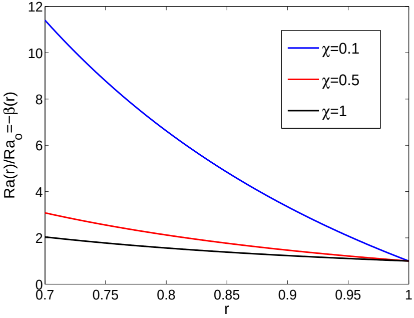
-

$$\beta(r) = \frac{\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr}}{\left[\left|\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr}\right|\right]_{r=r_o}} = -\frac{\mathrm{Ra}(r)}{\mathrm{Ra}_o} \Rightarrow \boxed{\beta(r) = \frac{1-\chi-(1/r)^2}{\chi}},$$

where

$$\chi = \left[\left|\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr}\right|\right]_{r=r_o} \Big/ \left|\frac{dT_{\mathrm{rad}}}{dr}\right|_{r=r_o}$$

Profile of $\beta(r)$ - Model A



- In a Cartesian geometry $\beta = -1$.
- In a spherical shell $\chi = 1$ and $\beta = -1/r^2$ for liquids (i.e. when $dT_{\mathrm{ad}}/dr = 0$), so there is still the effect of sphericity.
- $dT_{\mathrm{ad}}/dr \neq 0$ enhances that effect (while χ becomes smaller).
- In a weakly compressible spherical shell, the Rayleigh function is NOT constant unlike in Rayleigh-Bénard convection.

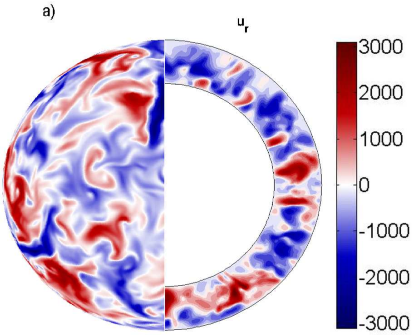
What is the effect of a varying $\beta(r)$ then?

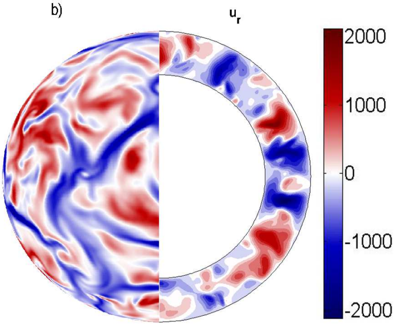
Numerical simulations

- 3D DNS solving the Boussinesq equations in a spherical shell with $r_i = 0.7$ and $r_o = 1$.
  (PARODY code)†
- Stress-free boundary conditions for the velocity.
- Fixed flux at the bottom:
  $$\frac{\partial \Theta}{\partial r} = 0|_{r=0.7}$$ and
  fixed temperature at the top: $\Theta = 0|_{r=1}$.
- $\mathrm{Ra}_o = 10^7$, Pr$= 0.1$

† Aubert, J., Aurnou, J., & Wicht, J. 2008, Geophysical Journal International, 172, 945
Dormy, E., Cardin, P., & Jault, D. 1998, Earth and Planetary Science Letters, 160, 15

Velocity slices snapshots





$\chi = 0.1$ $\chi = 0.5$

Kinetic Energy profiles

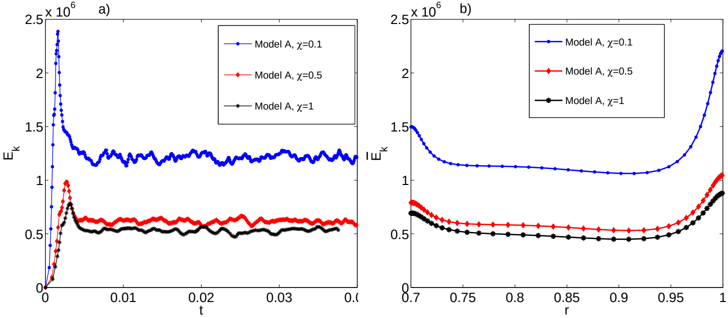
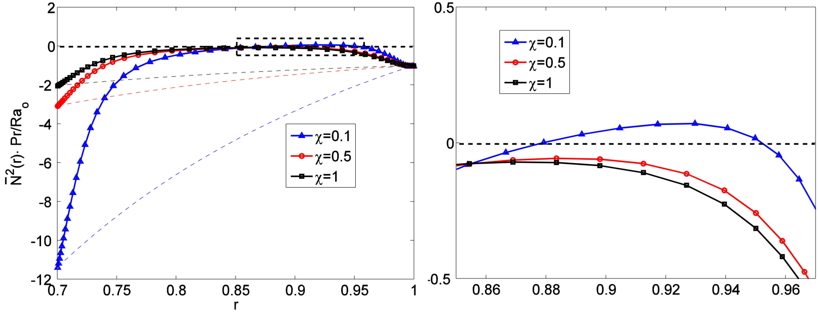


Figure: *a)* Kinetic energy plot with respect to time for $\mathrm{Ra}_o = 10^7$, Pr$= 0.1$ and three different χ. The system has reached a statistically steady state and it has thermally relaxed. *b)* Time-averaged kinetic energy for $\mathrm{Ra}_o = 10^7$ and Pr$= 0.1$.

Square of the non-dimensional buoyancy frequency profile

$\bar{N}^2(r) = (\beta(r) + d\bar{\Theta}/dr)\dfrac{\mathrm{Ra}_o}{\mathrm{Pr}}$ (solid lines) along with the radiative buoyancy frequency $N^2_{\mathrm{rad}}(r) = \beta(r)\dfrac{\mathrm{Ra}_o}{\mathrm{Pr}}$ (dashed lines)



1. What are the properties of this slightly subadiabatic region emerging close to the outer boundary?

2. Which of the physics elements lead to the subadiabatic layer?

   $\rightarrow$ Is it related to the varying Rayleigh function configuration of Model A?

Now, let's create a new model, Model B, where we have a constant Rayleigh function across the shell.

Spherical shell with a constant Rayleigh function

We can create a constant Rayleigh function across the shell by varying the thermal expansion coefficient $\alpha(r)/\alpha_o$ such that:

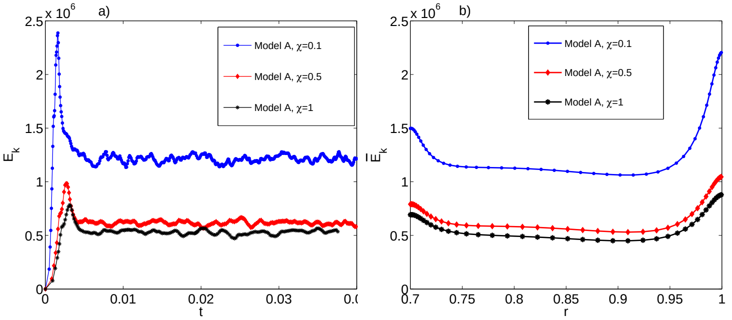
$$\mathrm{Ra}(r) = -\frac{\alpha(r) g \left(\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr} \right) r_o^4}{\kappa \nu} = -\mathrm{Ra}_o \cdot \frac{\alpha(r)}{\alpha_o} \cdot \beta(r) = \mathrm{Ra}_o,$$

as long as

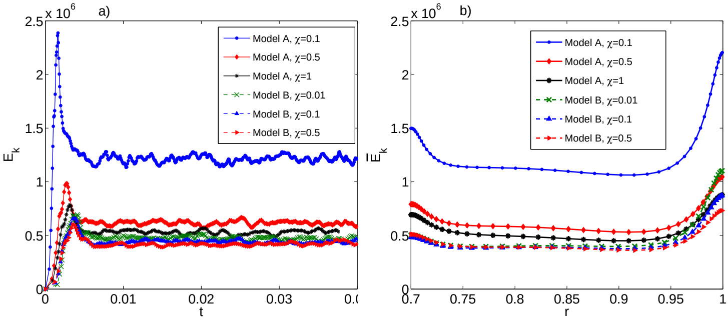
$$\frac{\alpha(r)}{\alpha_o} = -\frac{1}{\beta(r)},$$

where $\beta(r) = \dfrac{1 - \chi - (1/r^2)}{\chi}$ as in Model A.

KE profiles - Model A

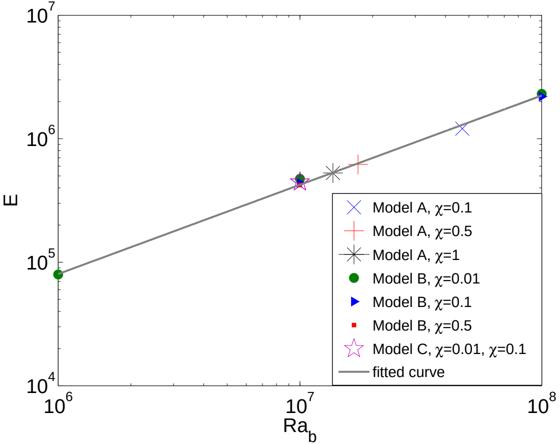


KE profiles - Model A and B



Mean kinetic energy E vs. bulk Rayleigh number Ra_b

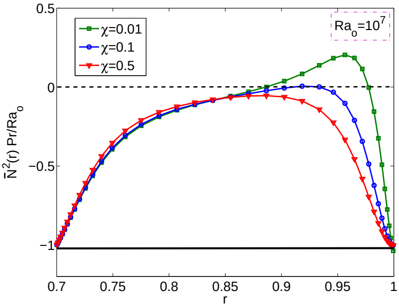
$$\boxed{E = C(\mathrm{Pr}, r_i/r_o)\mathrm{Ra}_b^{0.72}} \approx 3.7\mathrm{Ra}_b^{0.72},\ \mathrm{Ra}_b = \frac{\int_{r_i}^{r_o} \mathrm{Ra}(r) r^2 dr}{\int_{r_i}^{r_o} r^2 dr}$$

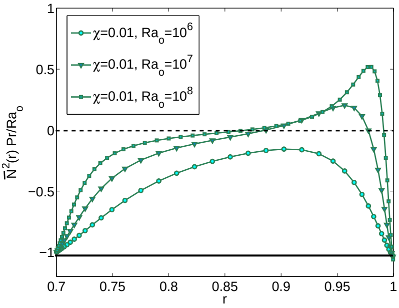


Square of the non-dimensional buoyancy frequency profiles

$$\bar{N}^2(r) = \frac{\alpha(r)}{\alpha_o}\left(\beta(r) + \frac{d\bar{\Theta}}{dr}\right)\frac{\mathrm{Ra}_o}{\mathrm{Pr}}$$ compared with the background

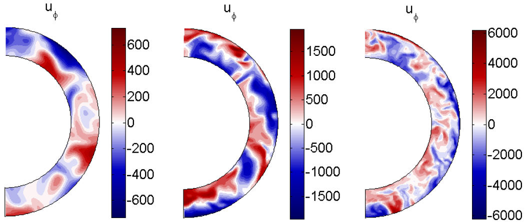
$$N^2_{\mathrm{rad}} = \frac{\alpha(r)}{\alpha_o}[\beta(r)]\frac{\mathrm{Ra}_o}{\mathrm{Pr}}\ ^{\ddagger}$$



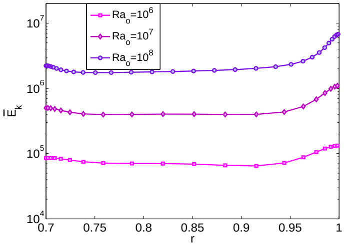


‡ In this setup all the simulations have the same background $N^2_{\mathrm{rad}}\mathrm{Pr}/\mathrm{Ra}_o = -1$ regardless of χ.

Velocity u_ϕ snapshots and kinetic energy for $\chi = 0.01$



$Ra_o=10^6$ $Ra_o=10^7$ $Ra_o=10^8$



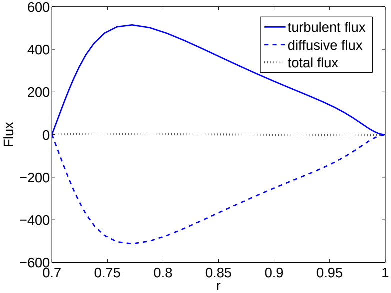
Convection is a very *non-local* process!!

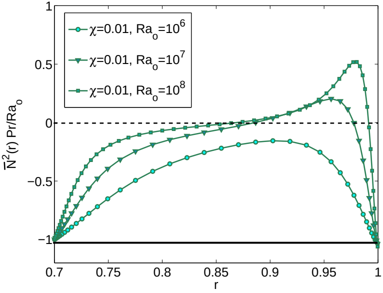
Results implicitly related to the choice of BCs on Θ:

- Flux at r_i: $d\Theta/dr|_{r_i} = 0 \Rightarrow$ Flux at r_o: $d\Theta/dr|_{r_o} = 0$ when the system is in thermal equilibrium.

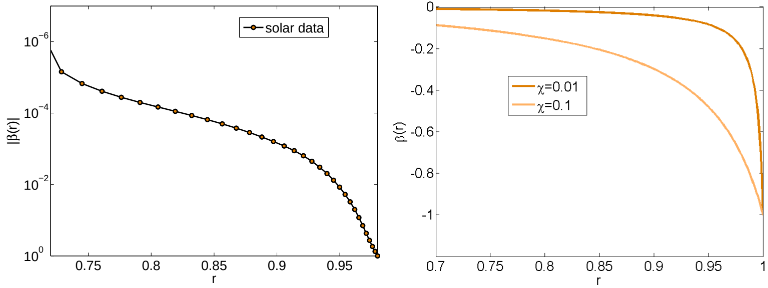
  $\rightsquigarrow$ $F_T = 0$ in equilibrium $\Rightarrow$ turbulent flux+diffusive flux$= 0 \Rightarrow$ $\bar{F}_{turb} = \dfrac{1}{\mathrm{Pr}}\dfrac{d\bar{\Theta}}{dr}$:

$$\bar{N}^2(r) = \frac{\alpha(r)}{\alpha_o}[\beta(r) + d\bar{\Theta}/dr]\frac{\mathrm{Ra}_o}{\mathrm{Pr}} = \frac{\alpha(r)}{\alpha_o}\boxed{[\beta(r) + \mathrm{Pr}\bar{F}_{turb}]}\frac{\mathrm{Ra}_o}{\mathrm{Pr}}$$



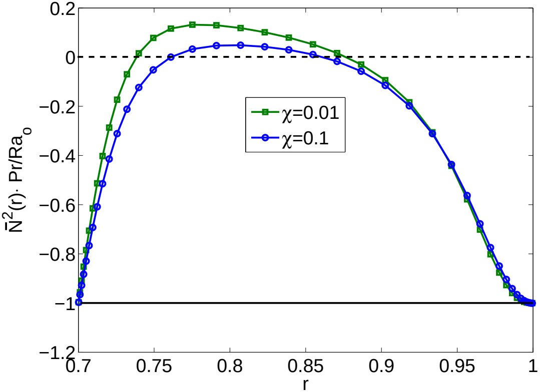


Solar-like $\beta(r)$ profile



$\bar{N}^2(r)\mathrm{Pr}/\mathrm{Ra}_o$ profile for $\mathrm{Ra}_o = 10^7$

subadiabatic layer still exists, now closer to the inner boundary!



Summary

- The mean kinetic energy depends solely on the bulk Ra_b such that $E \propto \mathrm{Ra}_b^{0.72}$.
- Emergence of subadiabatic region due to:
  1. mixed temperature boundary conditions,
  2. sufficiently turbulent flows (high Ra),

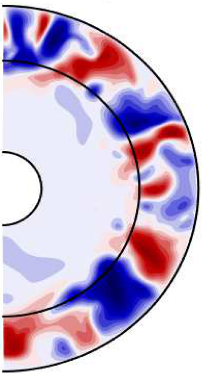
  and enhanced by:

  3. large superadiabaticity contrast (i.e. strongly varying $\beta(r)$ (low χ)).
- Convection vigorous everywhere: highly non-local convection!

Part II

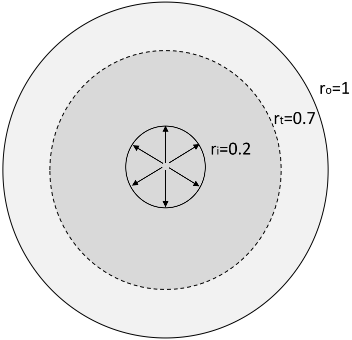
Convective overshooting and penetration in a spherical shell

Overshooting and penetrative convection



- In solar-like stars the bottom of the CZ is not impermeable but instead it sits on top of a stable RZ.
- Convective eddies can propagate into the RZ through inertia, which is commonly referred to as *overshooting*.
- This can cause both chemical and thermal mixing.
- Past studies distinguish between two regimes:
  1) overshooting: plumes only mix chemical species
  2) penetrative: the effect is so strong as to extend the CZ (beyond what linear theory predicts).

Spherical shell and BCs



- fixed flux at the inner boundary at $r_i = 0.2$
- fixed temperature at the outer boundary at $r_o = 1$
- CZ-RZ interface located at $r_t = 0.7$

Non-dimensional Equations

The non-dimensional equations are as before:

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{\mathrm{Ra}_o}{\mathrm{Pr}} \Theta \boldsymbol{e_r} + \nabla^2 \boldsymbol{u},$$

and

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta + \boxed{\beta(r)} u_r = \frac{1}{\mathrm{Pr}} \nabla^2 \Theta,$$

where now $\beta(r)$ is chosen such that we have:

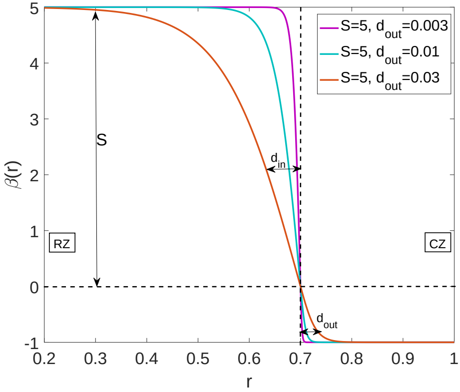
- a convectively stable RZ for $r < 0.7$

- a convectively unstable CZ for $r \geq 0.7$.

Profile of $\beta(r)$

stiffness parameter $\boxed{S}$: defines how stable the RZ is to convection.

transition width $\boxed{d_{out}}$: defines the steepness of the transition slope



$$\beta(r) = \begin{cases} -S \tanh\left(\dfrac{r - r_t}{d_{in}}\right), & r < r_t \\ -\tanh\left(\dfrac{r - r_t}{d_{out}}\right), & r \geq r_t \end{cases}$$

Non-dimensional quantities

- The Rayleigh number and the Rayleigh function

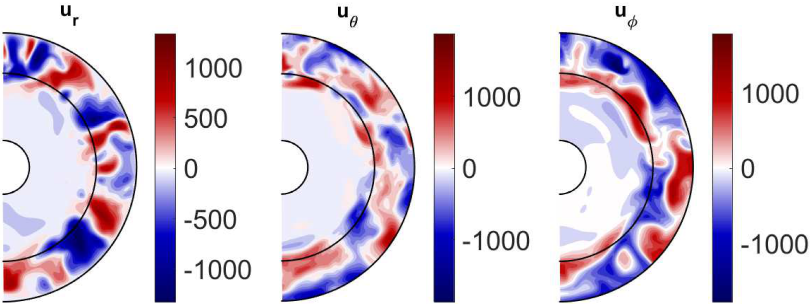
$$\mathrm{Ra}_o = \frac{\alpha g \left[\left| \frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr} \right| \right]_{r=r_o} r_o^4}{\nu\kappa}$$

$$\mathrm{Ra}(r) = -\frac{\alpha g \left(\frac{dT_{\mathrm{rad}}}{dr} - \frac{dT_{\mathrm{ad}}}{dr} \right) r_o^4}{\nu\kappa}$$

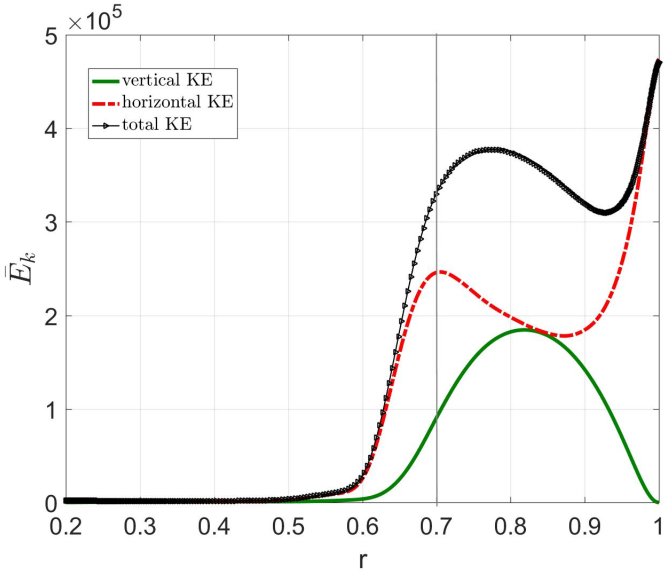
- Pr$= \nu/\kappa = 0.1$ for all the simulations.
- $\beta(r)$ is also $\beta(r) = -\mathrm{Ra}(r)/\mathrm{Ra}_o$.

Note: When Ra in the CZ increases, the RZ becomes more stable.

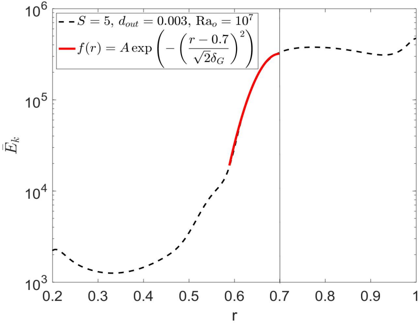
Meridional velocity snapshots for $S = 5$, $d_{out} = 0.003$ and $\mathrm{Ra}_o = 10^7$



KE for $S = 5$, $d_{out} = 0.003$ and $\mathrm{Ra}_o = 10^7$



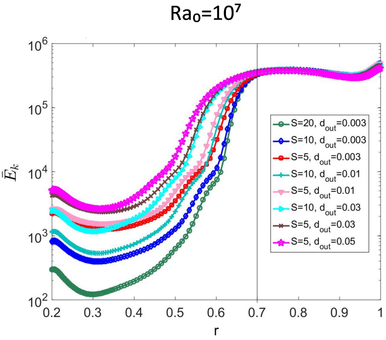
Log plot of $\bar{E}_k$

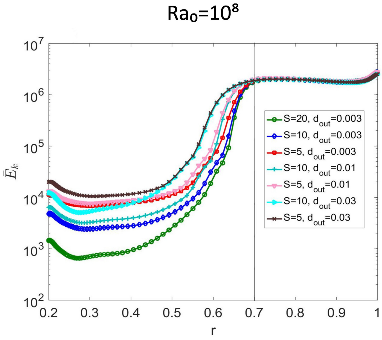


- Looks like a Gaussian below $r_t = 0.7$
- Gaussian fit function
  $$f(r) = A\exp\left(-\left(\frac{r-0.7}{\sqrt{2}\delta_G}\right)^2\right)$$
- A is the amplitude of the Gaussian.
- δ_G is the width of the Gaussian which gives a relative measure of how far the turbulent convective motions can on average travel into the stable RZ.

Kinetic Energies $\bar{E}_k$ for all the different input parameters

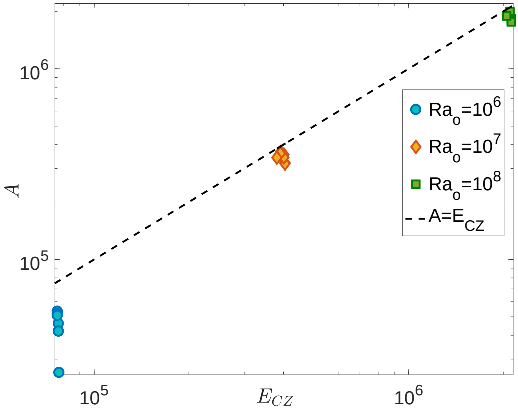
- mean kinetic energy in the CZ depends only on the bulk Ra_b.
- $\bar{E}_k$ scales like a Gaussian right below the bottom of the CZ for all the different simulations.





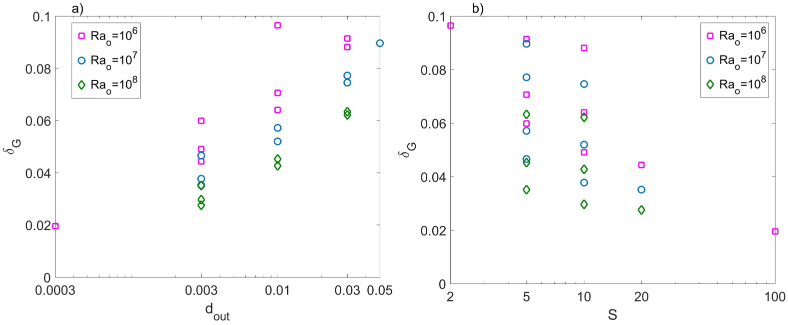
Prediction of the Gaussian amplitude A

$$\boxed{f(r) = A \exp\left(-\left(\frac{r-0.7}{\sqrt{2}\delta_G}\right)^2\right)},\ A \approx E_{CZ} = 3.7\mathrm{Ra}_\mathrm{b}^{0.72}$$



δ_G against d_{out} and S

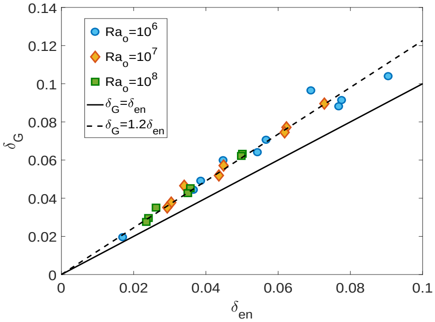
δ_G depends on the input parameters S, d_{out}, and Ra_o



But, can we also predict δ_G a priori?

Energetic argument for calculation of δ

Take a plume that starts from the base of the CZ with a mean KE of the CZ and travels inertially and adiabatically downward.
At the point at which Kinetic Energy=Potential energy, it will turn around!

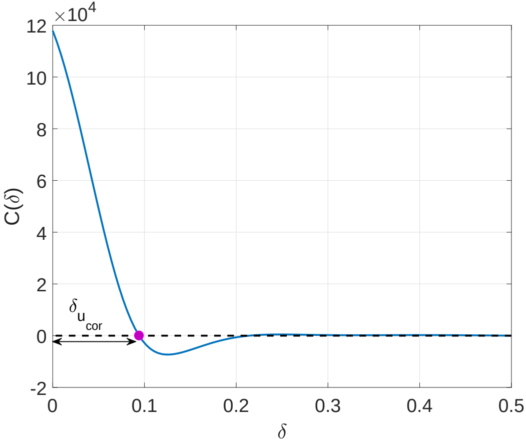


- Non-dimensionally, $E_{CZ} = -\dfrac{\mathrm{Ra}_o}{\mathrm{Pr}}\Theta\delta_{en}$
- But $E_{CZ} \approx 3.7\mathrm{Ra}_b^{0.72}$ and $\Theta \approx \Theta_{\mathrm{ad}}$.
- $3.7\mathrm{Ra}_b^{0.72} = \delta_{en}\, \dfrac{\mathrm{Ra}_o}{\mathrm{Pr}} \displaystyle\int_{0.7-\delta}^{0.7} \beta(r)dr$
- $\boxed{\delta_G = 1.2\delta_{en}}$

If the energetic argument is correct$\rightarrow$ any lengthscale will scale like δ_{en}!

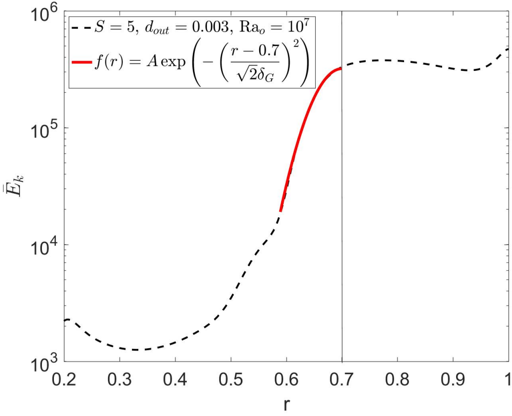
Auto-correlation function for the downflows

$$C(\delta) = \frac{1}{4\pi}\int_{t_1}^{t_2}\int_0^{2\pi}\int_0^{\pi} u_r(0.7,\theta,\phi)H(-u_r(0.7,\theta,\phi))u_r(0.7-\delta,\theta,\phi)\sin\theta d\theta d\phi dt$$



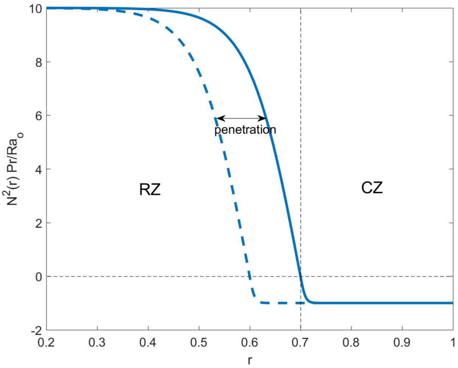
Back to $\bar{E}_k$

- The Gaussian part of $\bar{E}_k$ stops where $\delta_{u_{cor}}$ is defined!
- After that point, $\bar{E}_k$ decays exponentially.



Penetrative convection

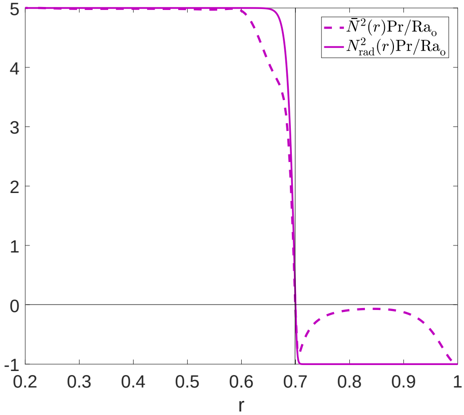
Standard cartoon picture of penetrative convection



- change of thermal stratification in the RZ, and
- extension of the CZ into the RZ

Is convection penetrative?

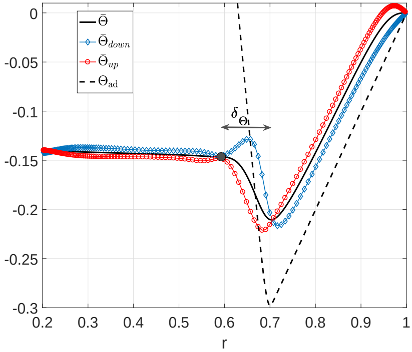
$S = 5$, $d_{out} = 0.003$ and $\mathrm{Ra}_o = 10^7$



No penetration... But there is partial **thermal mixing** in the RZ!

Temperatures for $S = 5$, $d_{out} = 0.003$ and $\mathrm{Ra}_o = 10^7$

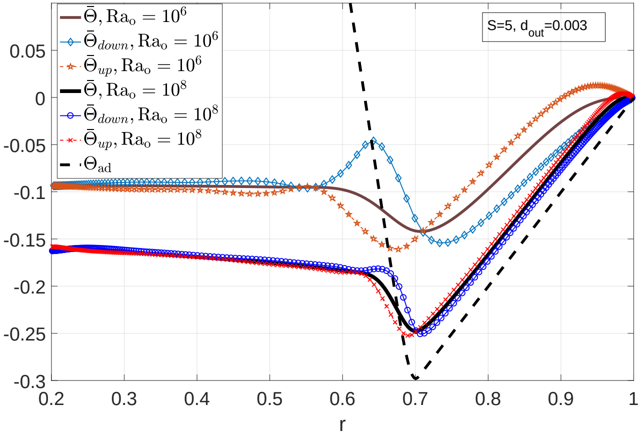
$\bar{\Theta}_{down}$: mean temperature of the downflows
$\bar{\Theta}_{up}$: mean temperature of the upflows
Θ_{ad} : adiabatic temperature



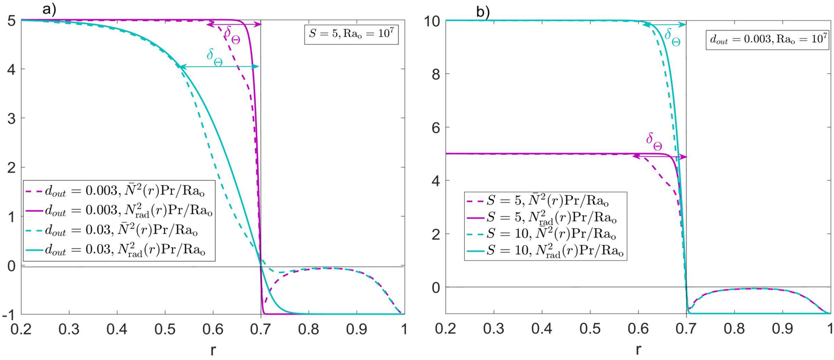
- Downflows carry cold material downward.
- They heat up while in the RZ due to adiabatic compression.
- Then they decelerate and match the mean temperature.
- Upflows have the exact opposite behavior.

δ_{Θ} gives a new lengthscale for thermal mixing!

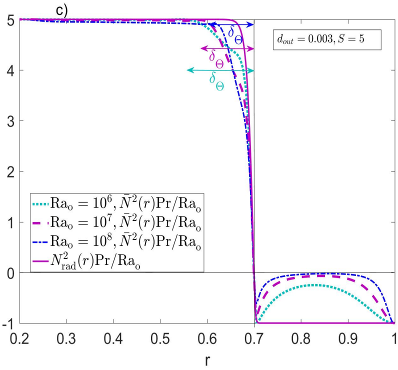
Temperatures for $S = 5$, $d_{out} = 0.003$



Thermal mixing in the RZ

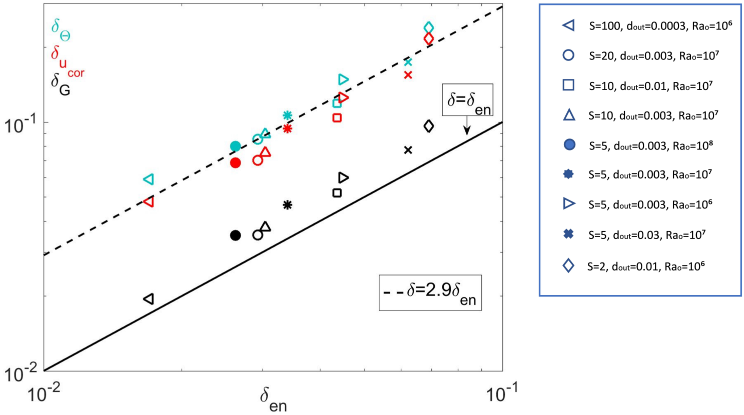


Thermal mixing in the RZ



- With higher Ra_o, the thermal mixing is shallower but more efficient!
- If we then increased Ra_o, could we finally see pure penetration?

Comparison of the different lengthscales



All the different lengthscales scale well with δ_{en}!

Conclusions

- No pure penetration, but not just overshooting either:

  $\rightsquigarrow$ Intermediate regime where there is partial thermal mixing in the RZ!

- The kinetic energy scales like a Gaussian below $r_t = 0.7$.

  $\rightsquigarrow$ We can actually model that region!

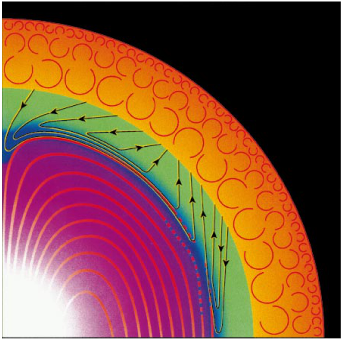
- All the different lengthscales scale well with δ_{en}.

  $\rightsquigarrow$ Then, we can predict $\delta_{\Theta,u_{cor}} \approx 3\delta_{en}$, and $\delta_G = 1.2\delta_{en}$.

Future goals

Models of the interior of the Sun rely on having a primordial magnetic field in the RZ.

Figure from Gough & McIntyre (1998) paper



- Add magnetic field in the RZ.
- Study the interaction of the field with the turbulent motions:

1. Can the field confine the overshooting motions from going deeper in the RZ?
2. Can these motions halt the magnetic field from diffusing outward into the CZ?

Thank you for your attention!

Questions?

...Extra slides...

Model	χ	Ra_o	N_r	N_θ	N_ϕ
(a)	0.1	10^7	250	402	480
(a)	0.5	10^7	220	346	384
(a)	1	10^7	220	346	384
(b)	0.01	10^6	200	192	192
(b)	0.01	10^7	200	288	320
(b)	0.01	10^8	300	516	640
(b)	0.1	10^7	200	288	320
(b)	0.1	10^8	300	516	640
(b)	0.5	10^7	200	288	320
(c)	0.01	10^7	200	288	320
(c)	0.1	10^7	200	288	320

$$\kappa \nabla^2 T_{\rm rad} = -H(r). \qquad (1)$$

BCs:

$$-\kappa \frac{dT_{\rm rad}}{dr}\bigg|_{r=r_i} = F_{\rm rad}, \quad T(r_o) = T_o. \qquad (2)$$

Integrating equation (1) once yields

$$\kappa \frac{dT_{\rm rad}}{dr} + F_{\rm rad} = -\int_{r_i}^{r} H dr, \qquad (3)$$

hence we can generate any functional form we desire for $dT_{\rm rad}/dr$ with a suitable choice of $H(r)$.